EFFECT OF VARIABLE THERMAL CONDUCTIVITY AND HEAT TRANSFER COEFFICIENT ON PERFORMANCE OF FINS

By

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DEPARTMENT OF MECHANICAL ENGINEERING

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EFFECT OF VARIABLE THERMAL CONDUCTIVITY AND HEAT TRANSFER COEFFICIENT ON PERFORMANCE OF FINS

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CERTIFICATE

This is to certify that the work on "EFFECT OF VARIABLE THERMAL CONDUCTIVITY AND HEAT TRANSFER COEFFICIENT ON PERFORMANCE OF FINS", has been carried out under my supervision and has not been submitted elsewhere for a degree.

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-SHYAM

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$A(x)$	Cross-sectional area of fin normal to x at point x
B	Non-dimensional parameter given by $L\sqrt{h_a/K_o y_o}$
C	A constant (Eq.[5.2])
D	A constant (Eq.[5.2])
$f(x)$	A function (Eq.[5.2])
h	Heat transfer coefficient
h_a	Average heat transfer coefficient
h_{eq}	Equivalent heat transfer coefficient
$h(x)$	Heat transfer coefficient at point x
K	Thermal conductivity
K_o	Thermal conductivity at ambient temperature
$K(\theta)$	Thermal conductivity at temperature excess, θ
L	Length of fin
m	A non-dimensional number given by $L^2 h_a/K_o y_o$
n	A constant
P	Perimeter of fin
$P(x)$	Perimeter of fin at point x
Q	Heat flux at point x
S	Surface area of fin between origin and point x
T	Temperature
T_a	Ambient temperature
u	A function defined in Chapter 5
W	Width of rectangular fin
x	Distance along axis normal to basic surface
x'	Non-dimensional distance along x given by x/L
y	Half thickness of fin at point x

y_o Half thickness of fin at fin-root

α Temperature coefficient of thermal conductivity

α' Non-dimensional temperature coefficient of thermal conductivity given by $\alpha\,\theta_o$

β Temperature coefficient of thermal conductivity

β' Non-dimensional temperature coefficient of thermal conductivity given by $\beta\theta_o^2$

η Efficiency of fin

ν Power of x' in expression for h(x)

θ Temperature excess

θ_o Temperature excess at fin-root

θ' Non-dimensional temperature excess given by θ/θ_o

$\theta'(0)$ Value of θ' at $x' = 0$

$d\theta'(1)/dx'$ Value of $d\theta'/dx'$ at $x' = 1$

$d\theta'(0)/dx'$ Value of $d\theta'/dx'$ at $x' = 0$

SYNOPSIS

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EFFECT OF VARIABLE THERMAL CONDUCTIVITY AND HEAT TRANSFER COEFFICIENT ON PERFORMANCE OF FINS

This work presents a theoretical investigation into the effect of variable thermal conductivity and heat transfer coefficient on the excess temperature distributions, temperature gradients and the efficiencies of fins of three different shapes. A general differential equation governing the heat transfer by conduction and convection in fins is derived. A parabolic expression for the variation of the temperature dependent thermal conductivity and an experimentally verified power series type of an expression for the heat transfer coefficient have been assumed. Results have been compared with those of the simple case, in which both the thermal conductivity and the heat transfer coefficient are held constant. The results reveal that variable thermal properties exercise a significant effect on the performance of the fins considered.

CHAPTER 1

INTRODUCTION

1.1 INTRODUCTION

In a conventional heat exchanger heat is transferred from one fluid to another through a metallic wall and, other things being equal, the rate of heat flow is directly proportional to the extent of the wall surface and to the temperature difference between one fluid and the adjacent surface. If thin strips of metal are attached to the basic surface, extending into one of the fluids, the total surface is thereby increased which results in an enhanced rate of heat flow. These attached heat conducting strips constitute what are generally termed "extended surfaces" or "fins". Fins are of various types, the most common among them being (a) Rectangular, (2) Annular, and (3) Spines.

The exact analysis of the heat transfer process in fins is a complicated problem. Analytical solutions have, however, been obtained after invoking a set of simplifying assumptions, essentially put forward by Murray [1] and Gardner [2], which posit thermal properties namely the thermal conductivity and the heat transfer coefficient as constant. However both of these assumptions are open to serious questions. Temperature dependence of thermal conductivity is sometimes not only strong but also bizarre [3].

There is also evidence suggesting a rapid increase in the magnitude of heat transfer coefficient towards the end of a fin and its variation from point to point along the length of the fin [4-7]. Against this background an attempt at a more rigorous analysis aimed at ascertaining the impact of these variable properties on fin performance seems to be in order.

1.2 LITERATURE REVIEW

Although the literature pertaining to the study of fins dates back to the work of Ingehouse [8] in 1789, most of the work since has been concentrated on convecting-fins with constant thermal properties. It is only lately that the effect of variable thermal properties on the performance of fins, whether convecting, radiating, or both, has been studied and shown to be quite significant particularly when large temperature difference exists. Cobble [9] in 1963 found the temperature distribution of a fin transferring heat by combined convection and radiation. He obtained the solution in terms of Jacobian elliptical functions with the aid of the Gregory-Newton forward interpolation formula. Stockman and Kramer [10], in the same year, presented the effects of a linearly varying conductivity and emissivity on the heat transfer of purely radiating fins, using the Runga-Kutta method. Hung and Appl [11]

analyzed fins with simultaneous surface convection and radiation and presented the temperature distribution for variable thermal properties and heat generation. Kosarev and Nevrovskii [12] considered a purely convecting-fin with temperature dependent thermal conductivity and developed the method of presenting the tip temperature and the heat transfer rate. A regular perturbation solution for a straight convecting fin of constant thickness with temperature dependent conductivity has been presented by Aziz and Huq [13]. More recently Razelos and Imre [14] found optimum dimensions of circular fins of trapezoidal profile with variable thermal conductivity and heat transfer coefficient . Because of the complex nature of the governing differential equation, most of the solutions are numerical.

1.3 SCOPE OF THE PRESENT WORK

The literature review reveals that not much work has been done on fins with variable thermal properties. Whatever little work has been done, it confines itself to the study on fin performance of only one thermal property at a time, either the variable thermal conductivity or the variable heat transfer coefficient. Proper knowledge of the effect of each of these properties separately on fin performance is, no doubt, quite essential for accurate design

purposes. Since in actual practice both thermal conductivity and heat transfer coefficient come into play simultaneously, it is but natural to investigate the combined effect of these two variables on fin efficiency.

Also in most of the papers cited above temperature dependence of thermal conductivity has been assumed to be linear. A survey of the literature on thermal conductivity of metals reveals that a parabolic variation is not only more in order but it also accurately reflects the nature of the thermal conductivity of metals over a large range of temperatures. Keeping this in view, analyses have been made to predict both the individual and the combined effects of these thermal properties on fin-solutions. A parabolic variation for the thermal conductivity and a power series function for the variable heat transfer coefficient has been assumed.

CHAPTER 2

PROBLEM FORMULATION AND SOLUTION

2.1 STATEMENT OF THE PROBLEM AND THE ASSUMPTIONS INVOLVED

Accurate knowledge of temperature variation and heat flux along the length of a fin is imperative for its proper design. The governing heat transfer differential equation for a fin has been solved by many authors with certain simplifying assumptions. The work presented herein entails the solution of the governing equation with some of the assumptions relaxed and the comparison of results with the conventional solution available from literature.

The conventional mathematical analyses are normally based upon the following assumptions credited to Murray [1] and Gardner [2] .

1. Heat flow is steady
2. Fin material is homogeneous and isotropic
3. There are no heat sources or sinks in the fin itself
4. Heat flow to or from the fin surface at any point is directly proportional to the temperature difference between the surface at that point and the surrounding fluid
5. Thermal conductivity of the fin material is constant

6. Heat transfer coefficient is the same all over the fin surface
7. Temperature of the surrounding fluid is uniform
8. Temperature at the base of the fin is uniform
9. Fin thickness is so small compared to its height that temperature gradients normal to the surface may be neglected
10. Heat transferred through the outermost edge of the fin is negligible compared to that passing through the sides.

2.2 RELAXATION OF SOME OF THE ASSUMPTIONS AND DERIVATION OF EQUATION

In the present analysis an attempt has been made to relax assumptions five and six given above. In forced convection the heat transfer coefficient is seen to vary all along the fin length. Due to complexities in fin-geometry, it becomes very difficult to predict the nature of this variation. However, attempts made in this direction [4-7] have shown its effect on the overall heat transfer from a fin to be significant. But so far no attempt has been made to study the simultaneous effect of variable thermal conductivity and heat transfer coefficient on the performance of a fin. It is common knowledge that thermal conductivity varies not only with the material but also with temperature. Mathematically, variable heat

transfer coefficient and thermal conductivity can be expressed as

$$h = h(x),$$

and

$$K = K\ (\theta)\ .$$

Next the general differential equation governing heat transfer in fins relaxing assumptions five, and six will be derived.

The differential equation for the fin profile shown in Fig. (2.1) is formulated from a consideration of the steady-state heat balance over a differential element of length dx. The difference between the heat entering the differential element by conduction at x and that leaving the differential element by conduction at x + dx is given by

$$dQ = \frac{d}{dx}\left[K(\theta)\ A(x)\ \frac{dT}{dx}\right] dx \qquad (2.1)$$

To satisfy the heat balance equation, an equal amount of heat should be dissipated by convection to the surrounding medium. Thus,

$$dQ = h(x)\ (T - T_a) dS \qquad (2.2)$$

where dS is the surface area of the fin between x and x + dx. Also,

$dS = P(x)\ dx$ which reduces to

$$dS = P\ dx$$

for a constant cross-sectional area fin. Here P is a constant unlike P(x) which is a function of x.

From Eqs. (2.1) and (2.2)

$$\frac{d}{dx}\left[K(\theta)\,A(x)\,\frac{dT}{dx}\right] dx - h(x)\,(T-T_a)\,dS = 0 \qquad (2.3)$$

or,

$$\frac{d}{dx}\left[K(\theta)\,A(x)\,\frac{d\theta}{dx}\right] - h(x)\,\theta\,\frac{dS}{dx} = 0$$

or,

$$K(\theta)\,A(x)\,\frac{d^2\theta}{dx^2} + A(x)\,\frac{d\theta}{dx}\,\frac{dK(\theta)}{dx} + K(\theta)\,\frac{d\theta}{dx}\,\frac{dA(x)}{dx} - h(x)\,\theta\,\frac{dS}{dx} = 0$$

or,

$$\frac{d^2\theta}{dx^2} + \frac{1}{K(\theta)}\,\frac{d\theta}{dx}\,\frac{dK(\theta)}{dx} + \frac{1}{A(x')}\,\frac{d\theta}{dx}\,\frac{dA(x)}{dx} - \frac{h(x)}{K(\theta)\,A(x)}\,\frac{dS}{dx}\,\theta = 0$$

In order to non-dimensionalize the above equation, we put

$$\theta' = \theta/\theta_o,$$

and

$$x' = x/L.$$

This gives

$$\frac{d^2\theta'}{dx'^2} + \frac{1}{K(\theta')}\,\frac{d\theta'}{dx'}\,\frac{dK(\theta')}{dx'} + \frac{1}{A(x')}\,\frac{d\theta'}{dx'}\,\frac{dA(x')}{dx'} - \frac{L^2 h(x')}{K(\theta')A(x')}\,\frac{dS}{dx}\,\theta' = 0 \qquad (2.4)$$

Since our primary interest lies in studying how strongly the variable thermal properties affect the fin solution, the general term of variable cross-sectional area in the above equation could be dropped. Eq.(2.4) then reduces to

$$\frac{d^2\theta'}{dx'^2} + \frac{1}{K(\theta')}\frac{d\theta'}{dx'}\frac{dK(\theta')}{dx'} - \frac{L^2 h(x')P}{K(\theta')\,A}\theta' = 0 \qquad (2.5)$$

The variation of thermal conductivity of metals with temperature is highly irregular and different for different materials. For most of the materials, the variation is more pronounced at low temperatures, and becomes moderate around room temperatures and above, as in the case of iron and its alloys, or even negligible as in the case of aluminium. Quite cumbersome expressions for thermal conductivity, varying in nature from material to material, are available for quite a few materials, which describe its behaviour over a large range of temperatures. But fortunately the behaviour of some of the widely used fin materials like copper, aluminium and iron is smooth at and above room temperature and can be very well represented by the general parabolic curve given by the following expression

$$K(\theta) = K_o\,(1 + \alpha\theta + \beta\ \theta^2) \qquad (2.6)$$

where K_o is the thermal conductivity at room temperature and α and β are the temperature coefficients of thermal

conductivity. In the light of the above, it becomes imperative to use the general expression for the thermal conductivity given by Eq.(2.6) as against holding it constant. Constant thermal conductivity assumption may not only introduce significant errors into the solution but also restrict its scope. Eq.(2.6) can also be written as

$$K(\theta) = K_o \left(1 + \alpha\,\theta_o\,(\theta/\theta_o) + \beta\theta_o^2\,(\theta/\theta_o)^2\right)$$

or

$$K(\theta') = K_o(1 + \alpha'\,\theta' + \beta'\,\theta'^2) \tag{2.7}$$

where α' and β' are the non-dimensional temperature coefficients of thermal conductivity

$$\alpha' = \alpha\,\theta_o,$$

and

$$\beta' = \beta\,\theta_o^2\,.$$

Substitution of Eq.(2.7) in Eq.(2.5) yields

$$\frac{d^2\theta'}{dx'^2} + \frac{1}{K_o(1+\alpha'\,\theta' + \beta'\,\theta'^2)}\,\frac{d\theta'}{dx'}\,\frac{d}{dx'}K_o(1+\alpha'\,\theta' + \beta'\,\theta'^2)$$

$$- \frac{L^2 h(x')\,P}{K_o(1+\alpha'\,\theta' + \beta'\,\theta'^2)A}\,\theta' = 0$$

or,

$$\frac{d^2\theta'}{dx'^2} + \frac{\alpha'}{(1+\alpha'\theta' + \beta'\theta'^2)} \left(\frac{d\theta'}{dx'}\right)^2 + \frac{2\beta'\theta'}{(1+\alpha'\theta' + \beta'\theta'^2)} \left(\frac{d\theta'}{dx'}\right)^2$$

$$- \left(\frac{L^2 h(x')P}{K_o A}\right) \frac{\theta'}{(1+\alpha'\theta'+\beta'\theta'^2)} = 0 \qquad (2.8)$$

If α' and β' are put equal to zero in the above equation, it reduces to the simple case of constant thermal conductivity, viz.

$$\frac{d^2\theta'}{dx'^2} - \left(\frac{L^2 h(x') P}{K_o A}\right) \theta' = 0 \qquad (2.9)$$

Comparison of Eqs.(2.8) and (2.9) reveals the presence of two additional terms in Eq.(2.8) viz. the second and the third which take into account the effect of temperature dependence of thermal conductivity. In order to ascertain the significance of these terms, an order of magnitude analysis has been resorted to. The analysis requires a fair knowledge of the approximate values of α' and β' in the range of temperatures involved. Table 2.1 below gives the approximate values of thermal conductivity at different temperatures for copper, iron, and austenitic steel [15] .

TABLE 2.1

VALUES OF THERMAL CONDUCTIVITY AT DIFFERENT TEMPERATURES IN (W/m°K)

Temperature (°K) / Material	100	200	300	400	500
Copper	370	410	400	390	370
Iron	100	85	75	65	55
Austenitic Steel	9.5	12.0	14.5	17.0	20.0

Curve fitting in the range of 300°-500°K yielded the following values of K_o, α and β for the above three materials (see Table 2.2).

TABLE 2.2

VALUES OF α, β AND K_o FOR DIFFERENT MATERIALS

Materials	K_o (W/m°K)	α(1/°K)	β (1/°K^2)
Copper	390	-1/8000	-1/80,0000
Iron	75	-1/750	0
Austenitic Stainless Steel	14.5	$\frac{1}{670}$	$\frac{1}{60,0000}$

If the maximum value of α ~~are~~ is taken to be 1/670 then α' comes out to be on the order of 0.3 for a temperature excess of 150°K. A purely parabolic temperature dependence of thermal conductivity would require β' also to be equal to 0.3 for the 30 percent increase in the value of thermal conductivity. The maximum value of θ' being 1, the typical values of coefficients for the second and the third terms viz. $\alpha'/(1 + \alpha' \theta' + \beta' \theta'^2)$ and $2\beta'\theta'/(1 + \alpha' \theta' + \beta'\theta'^2)$ comes out to be 0.23 and 0.46 respectively. Since the coefficient of the first term is 1, and the second and the third terms are also quite significant as computed above, they can not be ignored and must therefore be considered in the analysis.

VARIABLE HEAT TRANSFER COEFFICIENT

Literature on the behaviour of fins exposed to steady state surroundings with a variable heat transfer coefficient is meagre. The longitudinal fin of rectangular profile was studied by Han and Lefkowitz [4] and Chen and Zyskowski [5]. Experimental investigations of Ghai and Jacob [6] and Stachiewicz [7] suggest a power-series form of heat transfer coefficient for rectilinear fins. These papers confirm that for practical applications the assumption of a constant heat transfer coefficient is unrealistic and leads to erroneous results. Han and Lefkowitz recommend a

power-series form of $h(x')$ such as

$$h(x') = (\nu + 1) \; h_a \; (x')^{\nu} \tag{2.10}$$

where h_a is the average heat transfer coefficient with respect to the length of the fin. (It is to be noted here that average value of heat transfer coefficient with respect to heat dissipation, h_{eq}, is different from h_a. In this case h_{eq} turns out to be greater than h_a. For details please see Appendix C). The recommended values of ν by Han and Lefkowitz are one and two, which imply a linear and a parabolic variation of the heat transfer coefficient with the fin length respectively.

Chen and Zyskowski [5] assumed an exponential variation of the heat transfer coefficient given by

$$h(x) = h_a \; \frac{1 - a\, e^{-c\, x'}}{1 - a/c\, (1 - e^{-c})} \tag{2.11}$$

where a and c are constants.

Since the experimental results [6-7] confirm the power series form of a heat transfer coefficient given by Eq.(2.10), it is adopted here to study its impact on fin performance. Substitution of Eq.(2.10) into Eq.(2.8) yields

$$\frac{d^2\theta'}{dx'^2} + \frac{\alpha'}{(1+\alpha'\theta'+\beta'\theta'^2)}\left(\frac{d\theta'}{dx'}\right)^2 + \frac{2\beta'\theta'}{(1+\alpha'\theta'+\beta'\theta'^2)}\left(\frac{d\theta'}{dx'}\right)^2$$

$$-\left(\frac{L^2(\nu+1)h_aP}{K_o A}\right)\frac{\theta' x'^{\nu}}{(1+\alpha'\theta'+\beta'\theta'^2)} = 0 \qquad (2.12)$$

Boundary conditions: $\theta'(0) = 1$; $d\theta'(1)/dx' = 0$

where $0 \leq \theta' \leq 1$; $0 \leq x' \leq 1$

Now the whole problem lies in the solution of Eq.(2.12) with the specified boundary conditions.

2.3 SOLUTION FOR RECTANGULAR FIN

Variation of the heat transfer coefficient has been assumed as

$$h(x') = h_a(\nu+1)(x')^{\nu}$$

For $\nu = 0$ this expression reduces to

$$h(x) = h_a$$

and describes the constant heat transfer coefficient case. In order to make meaningful comparisons of results with the constant heat transfer coefficient case, average value of $h(x)$ over the fin length should be found out. Now

$$h_{av} = \int_0^1 h(x)dx' / \int_0^1 dx' = h_a$$

For rectangular fins

$$A = 2\,y_oW\text{, and}$$

$$P = 2W + 4y_o = 2W\,(1 + 2y_o/W)$$

$$\simeq 2W \quad \text{since } 2y_o/W \ll 1.$$

Let,

$$\frac{L^2 h_a P}{K_o\,A} = B^2 \tag{2.13}$$

where $B \overset{\text{def}}{=} L\sqrt{\frac{h_a}{K_o y_o}}$ is a non-dimensional number like Biot Number ($= h_aL/K_o$) and like it is proportional to the ratio of the internal resistance to heat flow by the fin material to the film resistance to heat transfer at the surface of the fin. The application of fins for heat dissipation in any system loses its meaning when Biot Number is greater than one because it results in lesser heat dissipation. But since $B = \sqrt{(\text{Biot No.} \times L/y_o)}$ and L/y_o is usually very large the upper limit of one does not hold good for the parameter B,.B cannot be infinitely large because other things being equal, the efficiency of fins goes down with increase in its magnitude and it, therefore, becomes logical to impose an upper limit on its values. In the present analysis the upper limit for B was fixed at 4 based upon the fin dimensions given by Kraus and Snider [16].

Substitution of B in Eq.(2.12) gives

$$\frac{d^2\theta'}{dx'^2} + \alpha'(d\theta'/dx')^2/(1+\alpha'\theta'+ \ '\theta'^2)+2\beta'\theta'(d\theta'/dx')^2/(1+\alpha'\theta'+\beta'\theta'^2)$$

$$- (\nu+1)\,B^2\,x'^{\nu}\,\theta'/(1+\alpha'\theta'+\beta'\theta'^2) = 0 \qquad (2.14)$$

Boundary conditions: $\theta'(0) = 1$; $d\theta'(1)/dx' = 0$.

This is a boundary value non-linear differential equation of the second order and because of its complex nature its analytical solution is hither to unknown. Power series method of solution also breaks down in this case as the resulting series turns out to be diverging in nature. Under the circumstances numerical solution remains the only way out. A considerable amount of literature is available on the numerical solution of differential equations. The details of the technique employed to solve the above equation and its justification as also the accuracy of the results obtained are discussed in Appendix-A. The solution comprises the values of θ' and $d\theta'/dx'$ at the grid points along the fin length, and the fin efficiency for a given set of parameters. The differential equation involves 4 parameters viz. α', β', ν and B and each set of values for these parameters results in a unique solution. The values of the parameters with their limits have been listed in Table 2.3 and the solutions have been obtained for all possible combination of the values therein.

TABLE 2.3

VALUES OF PARAMETERS

α'	-0.3, -0.2, -0.1, 0, 0.1, 0.2, 0.3,
β'	-0.3, -0.2, -0.1, 0, 0.1, 0.2, 0.3,
ν	0 , 1 , 2
B	0, 0.5, 1.0, 1.5, 2.0, 2.5, 3, 3.5, 4.0

For a fuller understanding of the fin and its performance, the concept of efficiency comes handy. It is defined as follows:

$$\eta = \frac{\text{Actual heat dissipation from a fin}}{\text{Heat dissipation from the fin if its entire surface is maintained at the fin root temperature}}$$

$$= \frac{Q}{Q_i} \qquad (2.15)$$

The heat dissipated from the fin is equal to heat flux at the fin-root. From Fourier's Law

$$Q = -K\,A\,\frac{dT}{dx} = -K\,A\,\frac{d\theta}{dx} = -\frac{KA\theta_o}{L}\,d\theta'/dx' \text{ at } x = 0 \qquad (2.16)$$

$$Q_i = h_a\,S(T_o - T_a) = h_a\,S\,\theta_o \qquad (2.17)$$

where h_a is the average value of the heat transfer coefficient.

From Eqs.(2.15), (2.16) and (2.17)

$$\eta = -\frac{K\,A\,\theta_o}{L\,h_a S\theta_o}(d\theta'/dx') = -\frac{K\,A}{L\,h_a S}(d\theta'/dx') \text{ at } x = 0 \qquad (2.18)$$

For rectangular fins: (Refer Fig.2.2(a))

$$\frac{K\,A}{L\,h_a S} = \frac{K\,2y_o W}{L\,h_a 2\,LW}$$

$$= \frac{K\,y_o}{h_a L^2} = K_o(1+\alpha'\theta'+\beta'\theta'^2)y_o/h_a L^2$$

$$= (1+\alpha'\theta'+\beta'\theta'^2)/B^2 \qquad (2.19)$$

From Eqs.(2.18) and (2.19)

$$\eta_{rect} = -(1+\alpha'\theta'+\beta'\theta'^2)(d\theta'/dx')/B^2 \text{ at } x = 0$$

$$= -(1+\alpha'+\beta')(d\theta'(0)/dx')/B^2 \qquad (2.20)$$

(Since $\theta'(0) = 1$)

Also, B, α', β' and $d\theta'/dx'$ at $x = 0$ being known, the η_{rect} can be readily computed.

2.4 SOLUTION FOR SPINES (Refer Fig.2.2(b))

For a circular spine of constant cross sectional area

$$\frac{L^2 h(x)P}{K\ A} = \frac{L^2(\nu+1)h_a x'^{\nu} 2\pi y_o}{K_o(1+\alpha'\theta' + \beta'\theta'^2)\pi y_o^2} = \frac{2\ L^2 h_a(\nu+1)\ x'^{\nu}}{K_o y_o\ (1+\alpha'\theta' + \beta'\theta'^2)} \tag{2.21}$$

Eqs.(2.12) and (2.21) give

$$d^2\theta'/dx'^2 + \alpha'(d\theta'/dx')^2/(1+\alpha'\theta'+\beta'\theta'^2)$$

$$+ 2\beta'\theta'(d\theta'/dx')^2/(1+\alpha'\theta'+\beta'\theta'^2)$$

$$-2B^2\ (\nu+1)\ x'^{\nu}\theta'/(1+\nu'\theta' + \beta'\theta'^2) = 0 \tag{2.22}$$

Boundary conditions: $\theta'(0) = 1$; $d\theta'(1)/dx' = 0$.

And,

$$\frac{K\ A}{h_a LS} = \frac{K_o(1+\alpha'\theta' +\beta'\theta'^2)\ y_o^2}{h_a\ L\ 2\ \ L\ y_o} = \frac{K_o y_o\ (1+\alpha'\theta' + \beta'\theta'^2)}{2\ h_a\ L^2}$$

$$= (1+\alpha'\ \theta' + \beta'\theta'^2)/2B^2$$

Substitution of the above expression in Eq.(2.18) results in

$$\eta_{spine} = -\ (1+\alpha'\ \theta'\ +\beta'\ \theta'^2)(d\theta'/dx')/2B^2 \quad \text{at } x = 0$$

$$= -(1\ +\alpha'\ +\beta')(d\theta'(0)/dx)/\ 2B^2 \qquad (2.23)$$

When Eq.(2.14) is compared with Eq.(2.22); and Eq.(2.20) with Eq.(2.23) it becomes obvious that they are similar except that $2B^2$ appears in place of B^2 in case of spines. This leads to the conclusion that the solution for both the cases will remain the same, except that the scale of B will be modified to $B/\sqrt{2}$ for spines. The same type of analysis can be extended to deal with the cases where cross-sections of spines are regular-polygons. All such cases involve just a change of scales, the magnitude of which depends upon the number of sides of the polygon.

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CHAPTER 3

RESULTS AND DISCUSSION

In the wake of the failure of analytical methods, numerical technique has been employed to solve the two-point boundary-value differential equation. Fourth Order Runga-Kutta method coupled with Shooting method has been preferred to other methods on account of its simplicity and accuracy. The division of non-dimensional length of the fin has been restricted to 10 grids only because further increase in the number of grids adds to accuracy only marginally. The values of non-dimensional temperature and its gradient have been determined at each grid point along with the efficiency of the fin using the expression given by Eq. (2.20) for a given set of parameters. Solutions have been obtained using the simplified assumption of the insulated fin end due to the following reasons:

1. In no way does the above assumption impede in our objective of studying the effect of variable thermal properties.
2. Inaccuracies introduced are insignificant.
3. It not only simplifies the boundary conditions but also eliminates Biot No. ($= h_a L/K_o$) parameter which, if present, would restrict the generality of the solution. The Biot No. arises from the real

boundary condition viz. $-Kd\theta/dx = h\theta$ at $x=L$ and it vanishes if the fin end is assumed to be insulated.

The FORTRAN program used to solve the equation satisfies the insulated end condition ($d\theta'/dx' = 0$) upto three places of decimal and can suitably be modified for further accuracies if the situation so warrants. To facilitate a better understanding of the results, it becomes necessary to group the variables appearing in the Eq.(2.14) and the subsequent analyses under the following two heads:

1. INDEPENDENT VARIABLES: (i) x', (ii) B, (iii) α', (iv) β', and (v) ν .
2. DEPENDENT VARIABLES: (i) θ', (ii) $d\theta'/dx'$, and (iii) η .

All the graphical results presented herein have been grouped into three categories, viz.

1. θ' vs. x' with α', β', ν, and B as parameters
2. $d\theta'/dx'$ vs. x' with α', β', ν, and B as parameters
3. η vs B with α', β', and ν as parameters.

Results falling under category 1 and 2 have been drawn on the same graph sheets to enable a clearer presentation and a better elucidation. This is so because θ' and $d\theta'/dx'$ are interlinked and nature of one can be readily

deduced from that of the other. Since the efficiency of the fin involves temperature gradient and the temperature-modified conductivity, a discussion of temperature and its gradient first will lead to a clear insight into the changes induced in the solution.

The results of the present investigations are presented as plots between various parameters from Figs.3.1 to 3.7. Figs.3.1(a) and 3.2(b) show the variation of non-dimensional temperature gradient and temperature with non-dimensional fin length for constant heat transfer coefficient condition and B = 1. The curves also demonstrate the effect of the temperature dependence of thermal conductivity. The increase in the tip-temperature is observed to be on the order of 17-18 percent for the condition $\alpha' = \beta' = 0.3$ as against the constant thermal conductivity case, $\alpha = \beta = 0$. This is only as expected because positive change in the values of parameters α' and β' result naturally in an increase in the average value of K which further tends to make the temperature distribution along the fin more or less uniform. The effect of α' ~~β'~~ on the variation of temperature is seen to be significant compared to the corresponding effect of β'. The reason for this lies in the nature of the relation of the variation of thermal conductivity with temperature viz.,

$$K(\theta) = K_o(1 + \alpha'\theta' + \beta'\theta'^2); \quad 0 \leq \theta' \leq 1$$

where θ' is the non-dimensional temperature. Because of the magnitude of the θ' in the above equation, ${\theta'}^2$ is always less than θ' and thus K increases more rapidly with α' than β'. In Fig.3.1(a) it is interesting to note that the temperature gradient at any given point decreases with increase in values of α' and β'. The decrease at the fin-root is on the order of 30 percent for $\alpha' = \beta' = 0.3$. This result is in consonance with the temperature distribution along the fin length. The higher values of thermal conductivity tend to bring about a uniform temperature distribution along the fin length which in turn results in a decrease of the temperature gradient at the fin-root. The small difference in the variation of temperature gradient observed in the curves for identical changes in α' and β' is also as expected and can be explained as before due to the variation of θ' from 0 to 1.

The variation of $d\theta'/dx'$ and x' can be further elucidated by considering the simple case of constant h and K. The expression for the temperature gradient in this case is (See Appendix B)

$$\frac{d\theta(0)}{dx'} = B \tanh B \text{ where } B = L\sqrt{\frac{h_a}{K_o y_o}}$$

In this equation both B and tanh B decrease with any increase in the value of K and since the magnitude of K decreases along the length of the fin the temperature

gradient registers a maximum drop at the fin root.

Figures 3.2(a) and (b) give the variation of non-dimensional temperature gradient and temperature with the length of the fin for both positive and negative values of α' for constant heat transfer coefficient conditions, B = 1, and $\beta = 0$, respectively. The temperature gradient is seen ? to increase while the temperature is observed to fall with crease ~~increase~~ in ~~length~~ α'. The effect of negative values of α' is much more pronounced in both the figures. This is as expected looking at the nature of the differential equation (2.12).

Figures 3.3(a) and 3.3(b) depict the variation of θ' and $d\theta'/dx'$ for various values of α', linear variation of heat transfer coefficient, and $\beta = 0$. The temperature gradient is seen to decrease at the fin-root by approximately 16-17 percent. The nature of the temperature gradient curve in this case is convex upward as against the concave upward shape of the curves for the simple case of constant heat transfer coefficient and thermal conductivity. Gradient changes from almost zero at the root to some finite value at the fin-tip. Since linear variation of h necessitates a lower heat dissipation near the fin-root and a higher heat loss towards the fin-tip, it explains the initially flat nature of the temperature gradient.

The marked drop in temperature gradient at the fin root can be explained by the fact that the heat transfer coefficient is zero at the fin-root and close to zero at the points near the fin root where the temperature is the highest. This results in smaller value of equivalent heat transfer coefficient (For details please see Appendix C).

To explain the sharp drop in temperature, heat transfer to and from the fin must be considered. Lower heat transfer to the fin, because of lower efficiency, results in higher temperature drop in this case whereas lower heat dissipation results in a correspondingly lower temperature drop. Both these opposite factors namely lower heat transfer into the fin and lower heat dissipation from the fin, are operating in this case on account of the smaller equivalent value of h. But the effect of the former, i.e. the lower heat transfer into the fin, seems to predominate thereby resulting in a higher temperature drop.

Figures 3.4(a) and 3.4(b) demonstrate the effect of ν on the temperature gradient and temperature distribution of a fin for B=1 and constant thermal conductivity conditions. With the increase in the value of ν both the temperature gradient and the temperature **excess** are seen to decrease **along the** fin. Changes seem to be more pronounced in the range of ν varying from 0 to 1. The temperature distribution curve seems to undergo a small change upto

about $x' = 0.4$ and thereafter difference for different values of ν seems to increase, which becomes maximum near the fin-tip. Similarly for the values of the temperature gradient, the slope near the fin-root seems to be zero for the values of $\nu = 1$ and 2 and thereafter it gradually increases towards the end of the fin. This is as expected because for values of $\nu = 1$ and 2 the heat dissipation at the root would be less in that order.

Figures 3.5(a) and 3.5 (b) exhibit clearly the combined effect of variable thermal conductivity and heat transfer coefficient on the temperature gradient and temperature distribution variation along the length of the fin for B = 1. In Fig.3.5(b) the effects of increase in the values of K and h seem to work at cross purposes with the result that the tip temperature seems to be the same for the conditions of $\alpha' = \beta' = .3, \nu = 2$ and $\alpha' = \beta' = 0, \nu = 0$. Otherwise the difference over the remaining length of the fin seems to be quite significant leading to the conclusion that higher values of ν and α and β lead to lower dissipation of heat from the fin.

Figures 3.6(a) through to 3.8(b) show the effect of B on the temperature gradient and temperature distribution variation along the fin length for constant thermal conductivity, constant heat transfer coefficient and variable heat transfer coefficient conditions, both linear and parabolic. B seems to

have quite a significant influence on both the temperature gradient and the temperature distribution. In all the three cases the end temperature seems to drop quite significantly. In Fig.3.6(b) the temperature excess at the fin tip is only 3 percent of that at the fin root when B = 4. Since large B values are equivalent to higher values of heat transfer coefficient, they naturally result in larger heat dissipation. This is demonstrated clearly in Fig.3.6(a) by the higher values of temperature gradient at the fin root. Lesser heat dissipation on account of smaller temperature excess towards the tip of the fin explains the change in the slope of the temperature gradient curve from higher to lower values along the length of the fin. For the case of ν = 1 (Figs.3.7(a), 3.7(b)) the temperature distribution curves follow the same pattern as in the previous case i.e. ν = 0 but the shapes of the curves representing temperature gradient are somewhat different being convex upward initially and changing gradually to concave upwards. In this case the heat transfer coefficient increases from a minimum value at the root of the fin to a maximum value at the fin-tip while θ' decreases from a maximum at the fin-root to a minimum at the fin tip. The maximum of the product h θ' which is proportional to the heat dissipation from the fin could lie anywhere in between the fin root and the tip. Since the heat dissipation is responsible for the change in the slope of the temperature gradient curves,

the above explanation seems to hold the key to the unusual behaviour of the curve. They seem to have the maxima of their slopes somewhere in the middle between the root and the tip. It is also interesting to note that for $\nu = 2$, the maximum value for the slope occurs at some intermediate point. Since heat flux at any point is given by

$$Q = - K \frac{d\theta}{dx} = - K_o (1 + \alpha\theta + \beta\theta^2) \frac{d\theta}{dx}$$

As Q decreases along the fin length because of continuous heat dissipation from the fin to the surroundings, so do θ and K also. However, if K decreases at a faster rate than Q, $\frac{d\theta}{dx}$ will obviously tend to increase in accordance with the dictates of the above relation.

Figures 3.9 to 3.14 give the variation of efficiency with B for different values of α', β' and ν. The efficiency which is a strong function of the parameter B, being cent percent at B = 0, decreases asymptotically to zero values as B tends to infinity.

Figure 3.9 shows the effect of variable conductivity on efficiency. Percentage change in efficiency increases very rapidly for small values of B but flatens towards the higher values of B. For the condition of $\alpha' = 0.3$, $\beta' = 0$ and $\nu = 0$ it starts from 0 at B = 0 and reaches

approximately 8.5 and 9.5 percent values for $B = 2$ and $B = 4$ respectively, while for $\alpha' = 0$, $\beta' = 0.3$ and $\nu = 0$, the percentage change in efficiency assumes a value of 6.4 only at $B = 4$, thereby establishing the dominance of α' over β'. This is in consonance with the pattern of results obtained and described previously.

Figure 3.10 brings out the relative importance of negative and positive coefficients of thermal conductivity. The negative temperature coefficient of thermal conductivity is seen to exercise a more pronounced effect on the fin efficiency in the negative direction compared to the positive temperature coefficient of thermal conductivity. For $\alpha' = 0.2$ and $B = 4$ the percentage change in efficiency is on the order of 6.5 while it is approximately 10.0 for $\alpha' = -0.2$ and $B = 4$. This can be explained as follows.

For $\beta' = 0$ and $\theta' = 1$ (non-dimensional temperature at the tip) the coefficient of the second term in differential Eq.(2.14) reduces to $\alpha'/(1+\alpha')$. Its values are 0.167 and 0.25 for $\alpha' = 0.2$ and -0.2 respectively. Since the magnitude of this coefficient viz, $\alpha'/(1+\alpha')$ represents the strength of the temperature dependence term of conductivity in the main differential equation, negative coefficient is bound to exercise a predominant influence on the general solution and the efficiency of the fin.

Figure 3.11 shows the effect of α' on efficiency for $\beta' = 0.2$ and $\nu = 0$. For $\alpha' = 0.3$ and $B = 4$ the percentage change in efficiency is on the order of 8.5 while it was 9.5 for $\alpha' = 0.3$, $\beta' = 0$ as seen in Fig.(3.9). This shows that any increase in the value of β' has the influence of reducing the effect α' on efficiency and vice-versa.

Figure 3.12 demonstrates the effect of α' on fin efficiency with $\nu = 1$, and $\beta' = 0$. The percentage change in η for the situation of $\alpha' = 0.3$, $B = 4.0$ and $\nu = 1$ is 12.5 while it is only 9.5 for $\alpha' = 0.3$, $B = 4.0$, and $\nu = 0$ as seem previously in Fig.(3.9). This shows that with variable heat transfer coefficient, the temperature dependent conductivity affects the performance of the fin more strongly and positively.

Figure 3.13 presents the effect of variable conductivity and heat transfer coefficient on fin performance. Holding the thermal conductivity constant a linear change of the heat transfer coefficient brings about a drop of approximately 40 percent in the efficiency, while a parabolic variation of h results in a higher drop of approximately 56 percent at $B = 4$. Again at $B = 4$, holding the heat transfer coefficient constant and varying the thermal conductivity, the augmentation in efficiency is seen to be only about 15.5 percent. This clearly shows the predominant effect of variable heat transfer coefficient on fin perfomance. If

both K and h are changed simultaneously, the figures for percentage change in the efficiency are seen to decrease to 30 and 45 for $\nu = 1$ and $\nu = 2$ respectively. This shows that both the variables h and K play a strong role in affecting the fin performance and none of them should therefore be neglected.

Lastly Fig.3.14 compares the efficiency of a rectangular fin and a circular spine. As explained before, the scale of B changes to $B/\sqrt{2}$ for a spine. Lower efficiency of spines for a given value of B is due to the fact that for a given cross sectional area and volume, the surface area of a cylinder is minimum. This results in lesser heat dissipation and hence smaller efficiency for the spines.

CHAPTER 4

CONCLUSIONS

From the results of the present analysis, the following inferences can be drawn:

(1) The temperature dependence of thermal conductivity, though being very small for Aluminium, is quite significant in the case of Iron, Copper and Austenitic Stainless Steel. Iron and copper being widely used fin materials, the study of variable thermal conductivity with its resultant impact on the performance of fins becomes imperative.

(2) Both increase and decrease of thermal conductivity of metals with temperature occur in practice depending upon the material and the range of temperatures involved. Temperature dependence of thermal conductivity around and above room temperature, for most of the materials, can be expressed by the following parabolic expression

$$K(\theta) = K_o (1 + \alpha\theta + \beta\theta^2)$$

(3) The rise in thermal conductivity with temperature results in

(a) a more uniform temperature distribution along the fin length

(b) a drop in the temperature gradient at the fin-root

(c) an increase in the efficiency of the fin.

The reverse of (a), (b), and (c) happens in case of a drop in the value of thermal conductivity with temperature.

(4) The percentage change brought about in the efficiency of a fin by a certain change in the value of its thermal conductivity is not proportional but less. For example, a 30 percent change in K results in an efficiency change of only 10 percent. Although less it is thought significant and therefore justifies the modified analysis. But all this is valid only if the parameter B is large (i.e. $B \geq 1$). For small values of B, K can be safely assumed to be constant without the risk of introducing any significant error into the results.

(5) Out of $\bar{\alpha}'$ and β' , the former is seen to exercise a stronger influence on the fin efficiency.

(6) The impact of variable h on fin performance is much more pronounced than that of K, and should be considered for all possible values of B. The assumption of variable h and constant K leads to significant errors and it is, therefore, recommended

that both be treated as variables for accurate results.

(7) The overall combined effect of variable h and K is seen to reduce the fin efficiency by as much as 65 percent in some cases and so a suitable safety factor needs to be employed for proper design purposes.

CHAPTER 5

SUGGESTIONS FOR FUTURE WORK

The limitation of the present analysis lies in its applicability only to fins with constant cross-sectional area. It also excludes from its purview the annular fins on account of their variable cross-sectional area. The present work can be extended on lines as suggested by Gardner [2] in order to bring under its range all possible fin shapes.

The author failed to pursue the problem along the suggested lines of Gardner partly because of lack of time and partly due to the complexity of the problem. However, for the benefit of future workers in this area, it is proper to list the difficulties involved and the method of attack of the problem.

The differential equation governing heat transfer in fins with variable conductivity, heat transfer coefficient and cross-sectional area is given as follows:

$$\frac{d^2\theta'}{dx'^2} + \left(1/A\,\frac{dA}{dx'}\right)\frac{d\theta'}{dx'} + \frac{\alpha'(d\theta'/dx')^2}{(1+\alpha'\theta'+\beta'\theta'^2)} + \frac{2\beta'\theta'}{(1+\alpha'\theta'+\beta'\theta'^2)}(d\theta'/dx')^2$$

$$- \left(\frac{L^2h(x)}{K_oA}\;\frac{dS}{dx}\right)\frac{\theta'}{(1+\alpha'\theta'+\beta'\theta'^2)} = 0 \qquad (5.1)$$

Evaluation of dA/dx involves simple calculus but $\frac{dS}{dx}$ poses a problem. $\frac{dS}{dx}$ can, however, be calculated from the following expression deduced by Gardner [2]

$$u = x\sqrt{\frac{h}{KA}\,\frac{dS}{dx}}$$

where u for different cases is given as follows.

1. STRAIGHT FINS: For variation of thickness like

$$y = y_o\ (x/L)^{\frac{1-2n}{1-n}}$$

The value of u is given as

$$u = 2(1-n)\ (x/L)^{\frac{1}{2(1-n)}}\ \sqrt{\frac{h}{Ky_o}}\ L$$

2. SPINES - CIRCULAR OR POLYGONAL SECTIONS:

The corresponding quantities in this case are given as

$$y = y_o\ (x/L)^{\frac{1-2n}{1-n}}$$

and

$$u = \frac{2\sqrt{3}\ (2-n)}{3}\ (x/L)^{\frac{3}{2(2-n)}}\ \sqrt{\frac{h}{Ky_o}}\ L$$

3. ANNULAR FINS:

$$y = y_o\ (x/L)^{\frac{-2n}{1-n}}$$

and

$$u = (1-n)\ (x/L)^{\frac{1}{1-n}}\ \sqrt{\frac{h}{Ky_o}}\ L$$

For n = 0, 1/2, 1, 3/2, and 2 substitution of expressions for $\frac{dA}{dx}$ and $\frac{dS}{dx}$ for different cases in Eq.(5.1) yields

$$\frac{d^2\theta'}{dx'^2} + \frac{C}{x'}\frac{d\theta'}{dx'} + \alpha'(d\theta'/dx')^2/(1+\alpha'\theta'+\beta'\theta'^2)$$

$$+2\beta'\theta'(d\theta'/dx')^2/(1+\alpha'\theta'+\beta'\theta'^2)$$

$$-\frac{DB^2 f(x)\,\theta'}{x^n\,(1+\alpha'\theta'+\beta'\theta'^2)} = 0 \qquad (5.2)$$

Boundary conditions: $\theta'(0) = 1$; $\frac{d\theta'(1)}{dx'} = 0.$?

$$0 \leq \theta' \leq 1 \; ; \quad 0 \leq x' \leq 1 \; ,$$

where C and D are constants; n = 0, 1/2, 1, 3/2 and 2,

$$h(x) = h_a\, f(x) \text{ and } B^2 = L^2 h_a/K_o y_o.$$

The presence of second and fifth terms in the above equation introduce a signularity at x = 0; and the whole problem boils down to one of solving a two-point boundary-value non-linear second order differential equation with a singularity. The singularity poses a big problem and many a numerical techniques break down in this case (See Appendix A). However, the problem involved is certainly not intractable and its analysis will go a long way in providing a highly generalized solution for fins of all shapes.

In the absence of experimental data on heat transfer coefficient for fins of different geometries, one has largely to depend upon theoretical models. A general solution

of the Eq.(5.2) with $h(x)$ based on experimental results should be attempted and a verification of the results thus obtained confirmed by subsequent experimental investigation into the performance of fins. Although a large amount of work is involved in the process, it is certainly worthwhile to pursue the same.

APPENDIX-A

The differential equation encountered in the present analysis is a two-point boundary-value non-linear second order equation. There are many numerical techniques to handle such problems e.g., Finite Difference Method, Least Square Method, Runga-Kutta Method etc. Since Runga-Kutta Method handles only the initial value problems, it has been coupled with Shooting Method to take care of the boundary-value problem. The methodology involves assuming a value for $d\theta'/dx'$ at the fin-root, solving the equation and checking whether the boundary condition at the fin-tip ($d\theta'/dx' = 0$) is satisfied. If not, then assuming a new value for the temperature gradient and thus keep on repeating the process till the boundary condition at the fin-tip is satisfied.

The method has been given preference over other ones on account of its simplicity and accuracy. The Fourth Order Runga-Kutta Method coupled with the satisfaction of the boundary-condition at the fin-tip upto three places of decimal gives an excellent accuracy. Table A-1 compares exact and numerically obtained values of non-dimensional temperature gradient, $d\theta'/dx'$, at the fin-root for constant h and K values.

TABLE A-1

B	0	1	2	3	4
Exact $\frac{d\theta'}{dx'}$ (A)	0	.76159	1.92806	2.98515	3.99732
Numerically obtained $\frac{d\theta'}{dx'}$ (B)	0	.76169	1.92852	2.98549	3.99746
Difference (B-A)	0.0	.00010	0.00046	0.00034	0.00014

Since changes of the order of 0.01 to .1 are involved, the accuracies obtained are more than sufficient.

APPENDIX-B

SOLUTION FOR FINS

The differential equation governing heat transfer in fins with constant heat transfer coefficient and thermal conductivity is given as

$$\frac{d^2\theta'}{dx'^2} - \frac{L^2 h_a P}{A\ K_o}\ \theta' = 0$$

$$\frac{d^2\theta'}{dx'^2} - m^2\ \theta' = 0$$

where,

$$m^2 = \frac{L^2 h_a P}{A\ K_o}$$

Boundary conditions $\theta'(0) = 1$, $\frac{d\theta'(1)}{dx} = 0$.

Solution of the above equation with its boundary conditions yields

$$\theta' = \text{Cos h m } (1 - x')/\text{Cos h m} \tag{B.1}$$

So,

$$\theta'(1) = \frac{1}{\text{Cos h m}}$$

Differentiation of Eq.(B.1) gives

$$d\theta'/dx' = -m \text{ Sinh h m } (1-x')/\text{Cos h m}$$

So,

$$\frac{d\theta'(0)}{dx'} = -m \tan \text{h m}$$

Now,

$$\eta = \frac{K_o A \ \theta_o (d \ \theta'(0)/dx')}{P \ L^2 \ \theta_o \ h_a} = \frac{K_o \ A}{P \ L^2 h_a} \frac{d\theta'(0)}{dx'} = \frac{1}{m^2} \frac{d\theta'(0)}{dx'}$$

$$= - \frac{m \ \tan h \ m}{m^2} \qquad = - \tanh m/m$$

For rectangular fins : $m = B$.

For spines (circular): $m = B\sqrt{2}$.

APPENDIX-C

EQUIVALENT HEAT TRANSFER COEFFICIENT

The form of the heat transfer coefficient suggested by Han and Lefkowitz [4] is

$$h = (\nu + 1) h_a x'^{\nu}$$

It's average value with respect to the fin length is

$$h_{av} = \int_0^1 (\nu + 1) h_a x'^{\nu} dx' = h_a .$$

But this value of average does not take into account the temperature variation along the fin-length.

It is logical to define equivalent heat transfer coefficient as that coefficient which, if assumed constant, would result in the same heat dissipation as the variable one.

Since, $Q = \int_0^1 h \, \theta (P \, dx')$.

We get,

$$\int_0^1 h_{eq} \, \theta \, P \, dx' = \int_0^1 h \, \theta' \, P \, dx'$$

or,

$$h_{eq} = \frac{\int_0^1 h \, \theta \, dx'}{\int_0^1 \theta' \, dx'} = \frac{h_a (\nu + 1) \int_0^1 x' \, \theta' \, dx'}{\int_0^1 \theta' \, dx'}$$

For $\nu = 1$.

$$h_{eq} = \frac{2h_a \int_0^1 x' \theta' dx'}{\int_0^1 \theta' dx'} = \frac{2h_a \int_0^1 \operatorname{Cos} h\, m(1-x')x'dx'}{\int_0^1 \cos h\, m\, (1-x')dx'}$$

$$= \frac{2h_a \dfrac{(\operatorname{Cos} hm-1)}{m^2}}{\dfrac{\sin h\, m}{m}}$$

$$= \frac{2h_a(\operatorname{Cos} h\, m-1)}{m \operatorname{Sin} h\, m}$$

The value of$\left(\frac{\operatorname{Cos} h\, m-1}{m\operatorname{Sin} h\, m}\right)$ is 1/2 at $m = 0$, and decreases continuously till it reaches zero at $m = \infty$. h_{eq}. is therefore, less than h_a except when $m = 0$.

```
      DIMENSION Y(100),S(2),E(2),EP(2),YP(100),C(15),D(15),D1(15)
     1,EF(15)
      G(X,Y,YP)=YP
      F(X,Y,YP)=C(IT1)*Y*(R+1.)*(X**R)/(1.+D(IT2)*Y+D1(IT3)*Y*Y)-(D(IT
     12)*YP*YP+2.*D1(IT3)*Y*YP*YP)/(1.+D(IT2)*Y+D1(IT3)*Y*Y)
      HB(X,YP)=YP
      READ*,A,B,BV1,BV2,N,NI,N1,N2,N3,R
      READ*,(C(I),I=1,N1)
      READ*,(D(I),I=1,N2)
      READ*,(D1(I),I=1,N3)
      DO 12 IT3=1,N3
      DO 11 IT2=1,N2
      DO 10 IT1=1,N1
      WRITE(21,101),C(IT1),D(IT2),D1(IT3)
  101 FORMAT(10X,'C=',F4.1,5X,'D=',F4.3,5X,'D1=',F4.3/)
      H=(B-A)/N
      M=N+1
      K=0
      L=0
      S(1)=0.
      S(2)=1.
    1 K=K+1
      L=L+1
      X=A
      Y(1)=BV1
      YP(1)=S(K)
      EF(1)=-(1.+D(IT2)+D1(IT3))*YP(1)/C(IT1)
      DO 2 I=1,N
      XK11=G(X,Y(I),YP(I))
      XK21=F(X,Y(I),YP(I))
      X=X+.5*H
      Z1=Y(I)+.5*H*XK11
      Z2=YP(I)+.5*H*XK21
      XK12=G(X,Z1,Z2)
      XK22=F(X,Z1,Z2)
      Z1=Y(I)+.5*H*XK12
      Z2=YP(I)+.5*H*XK22
      XK13=G(X,Z1,Z2)
      XK23=F(X,Z1,Z2)
      X=X+.5*H
      Z1=Y(I)+.5*H*XK13
      Z2=YP(I)+.5*H*XK23
      XK14=G(X,Z1,Z2)
      XK24=F(X,Z1,Z2)
      J=I+1
      Y(J)=Y(I)+H*(XK11+2.*XK12+2.*XK13+XK14)/6.
      EF(J)=-(1.+D(IT2)+D1(IT3))*YP(J)/C(IT1)
    2 YP(J)=YP(I)+H*(XK21+2.*XK22+2.*XK23+XK24)/6.
      E(K)=Y(J)
      EP(K)=YP(J)
      IF(BV2 .EQ. 0.) GO TO 3
      IF(ABS(1.-HB(E(K),EP(K))/BV2).LT.10.**(-3.)) GO TO 6
      GO TO 23
    3 IF(ABS(HB(E(K),EP(K))).LT.10.**(-3.)) GO TO 6
   23 IF(L.EQ.1) GO TO 1
      IF(L .GE. NI) GO TO 4
      IF(K .EQ. 2) K=K-2
      KK=K+1
      S(KK)=S(1)+(S(2)-S(1))*(BV2-HB(E(1),EP(1)))/(HB(E(2),EP(2))-HB
     1(E(1),EP(1)))
      GO TO 1
    4 WRITE(21,5) NI
    5 FORMAT(11X,'NO SOLUTION CAN BE OBTAINED USING THIS METHOD IN',I32
     1,'ITERATIONS'//11X,'THE BEST RESULT WAS')
    6 WRITE(21,7) L
    7 FORMAT(11X,'SOLUTION OBTAINED IN',I2,'ITERATIONS'//11X,'X',19X,'
     1Y',17X,'YP',19X,'EF')
      DO 8 I=1,M
      X=A+(I-1)*H
    8 WRITE(21,9) X,Y(I),YP(I),EF(I)
    9 FORMAT(4(2X,E18.9))
   10 CONTINUE
   11 CONTINUE
   12 CONTINUE
      STOP
      END
```

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